The variance is then

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 \Pi(x)\, dx = \int_{-1/2}^{1/2} x^2\, dx = \tfrac{1}{3}x^3\Big]_{-1/2}^{+1/2} = \tfrac{1}{12}\,,$$

perhaps not quite so obvious.

The normal distribution This whole lecture is about getting to Gaussians, so it seems appropriate that at some point I mention:

The Gaussian is a pdf.

Indeed, to borrow information from earlier work in this chapter, the Gaussian

$$g(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\,.$$

is a pdf with mean μ and variance σ^2. The distribution associated with such a Gaussian is called a *normal distribution*. There, it's official. But why is it "normal"? You're soon to find out.

3.6.5 Independence

An important extra property that random variables may have is *independence*. The plain English description of independence is that one event or measurement doesn't influence another event or measurement. Each flip of a coin, roll of a die, or measurement of a resistor is a new event, not influenced by previous events.

Operationally, independence implies that the probabilities multiply: If two random variables X_1 and X_2 are independent then

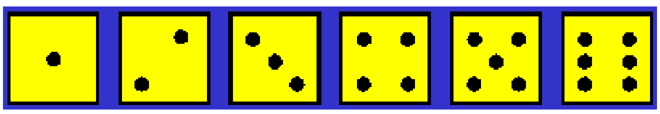
$$\text{Prob}(X_1 \le a \text{ and } X_2 \le b) = \text{Prob}(X_1 \le a) \cdot \text{Prob}(X_2 \le b)\,.$$

In words, if $X_1 \le a$ occurs r percent and $X_2 \le b$ occurs s percent then, if the events are independent, the percent that $X_1 \le a$ occurs *and* $X_2 \le b$ occurs is r percent of s percent, or rs percent.

3.6.6 Convolution appears

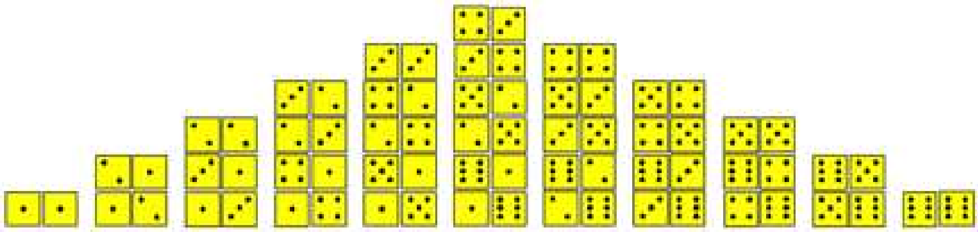
Using the terminology we've developed, we can begin to be more precise about the content of the Central Limit Theorem. That result — the ubiquity of the bell-shaped curve — has to do with *sums* of independent random variables and with the *distributions* of those sums.

While we'll work with continuous random variables, let's look at the discrete random variable $X =$ "roll the dice" as an example. The ideal histogram for the toss of a single die is uniform — each number 1 through 6 comes up with equal probability. We might represent it pictorially like this:



I don't mean to think just of a picture of dice here — I mean to think of the distribution as six bins of equal height 1/6, each bin corresponding to one of the six possible tosses.

What about the *sum* of the tosses of two dice? What is the distribution, theoretically, of the sums? The possible values of the sum are 2 through 12, but the values do not occur with equal probability. There's only one way of making 2 and one way of making 12, but there are more ways of making the other possible sums. In fact, 7 is the most probable sum, with six ways it can be achieved. We might represent the distribution for the sum of two dice pictorially like this:



It's triangular. Now let's see ... For the single random variable $X =$ "roll one die" we have a distribution like a rect function. For the sum, say random variables $X_1 + X_2 =$ "roll of die 1 plus roll of die 2", the distribution looks like the triangle function....

The key discovery is this:

> **Convolution and probability density functions** The probability density function of the sum of two independent random variables is the convolution of the probability density functions of each.

What a beautiful, elegant, and useful statement! Let's see why it works.

We can get a good intuitive sense of why this result might hold by looking again at the discrete case and at the example of tossing two dice. To ask about the distribution of the sum of two dice is to ask about the probabilities of particular numbers coming up, and these we can compute directly using the rules of probability. Take, for example, the probability that the sum is 7. Count the ways, distinguishing which throw is first:

$$\begin{aligned}
\text{Prob}(\text{Sum} = 7) &= \text{Prob}(\{1\text{ and }6\}\text{ or }\{2\text{ and }5\}\text{ or }\{3\text{ and }4\}\text{ or }\{4\text{ and }3\}\text{ or }\{5\text{ and }2\}\text{ or }\{6\text{ and }1\}) \\
&= \text{Prob}(1\text{ and }6) + \text{Prob}(2\text{ and }5) + \text{Prob}(3\text{ and }4) + \\
&\qquad \text{Prob}(4\text{ and }3) + \text{Prob}(5\text{ and }2) + \text{Prob}(6\text{ and }1) \\
&\qquad\quad \text{(probabilities add when events are mutually exclusive)} \\
&= \text{Prob}(1)\,\text{Prob}(6) + \text{Prob}(2)\,\text{Prob}(5) + \text{Prob}(3)\,\text{Prob}(4) + \\
&\qquad \text{Prob}(4)\,\text{Prob}(3) + \text{Prob}(5)\,\text{Prob}(2) + \text{Prob}(6)\,\text{Prob}(1) \\
&\qquad\quad \text{(probabilities multiply when events are independent)} \\
&= 6\left(\frac{1}{6}\right)^2 = \frac{1}{6}\,.
\end{aligned}$$

The particular answer, Prob(Sum = 7) = 1/6, is not important here[17] — it's the form of the expression

[17] But do note that it agrees with what we can observe from the graphic of the sum of two dice. We see that the total number of possibilities for two throws is 36 and that 7 comes up 6/36 = 1/6 of the time.